Semi-additive functionals and cocycles in the context of self-similarity *†‡

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Abstract

Self-similar symmetric α-stable, $\alpha \in (0,2)$, mixed moving averages can be related to nonsingular flows. By using this relation and the structure of the underlying flows, one can decompose self-similar mixed moving averages into distinct classes and then examine the processes in each of these classes separately. The relation between processes and flows involves semi-additive functionals. We establish a general result about semi-additive functionals related to cocycles, and identify the presence of a new semi-additive functional in the relation between processes and flows. This new functional is useful for finding the kernel function of self-similar mixed moving averages generated by a given flow. It also sheds new light on previous results on the subject.

1 Introduction

A symmetric α-stable ($S\alpha S$, in short), $\alpha \in (0,2)$, process $\{X(t)\}_{t\in\mathbb{R}}$ is called a (stationary increments) *mixed moving average* if it has an integral representation of the form

$$\{X_\alpha(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\int_X \int_{\mathbb{R}} \Big(G(x,t+u) - G(x,u)\Big) M_\alpha(dx,du)\right\}_{t\in\mathbb{R}} \tag{1.1}$$

or, equivalently, if the characteristic function of the process X_α is given by

$$E\exp\Big\{i\sum_{k=1}^{n}\theta_k X_\alpha(t_k)\Big\} = \exp\Big\{-\int_X\int_{\mathbb{R}}\Big|\sum_{k=1}^{n}\theta_k G_{t_k}(x,u)\Big|^{\alpha}\mu(dx)du\Big\}. \tag{1.2}$$

In these relations, $\stackrel{d}{=}$ denotes the equality in the sense of the finite-dimensional distributions, $(X,\mathcal{X},\mu)$ is a standard Lebesgue space, M_α is a $S\alpha S$ random measure on $X\times\mathbb{R}$ with control measure $\mu(dx)du$

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and $G : X \times \mathbb{R} \mapsto \mathbb{R}$ is a deterministic function, called a *kernel (function)*, such that the "time increment"

$$G_t(x,u) = G(x,t+u) - G(x,u), \quad x \in X, u \in \mathbb{R}, \tag{1.3}$$

satisfies $\{G_t\}_{t\in\mathbb{R}} \subset L^\alpha(X \times \mathbb{R}, \mu(dx)du)$. For more information on $S\alpha S$ random measures, processes and integral representations of the type (1.1), see for example Samorodnitsky and Taqqu (1994). Observe that, in view of (1.2), mixed moving averages have always stationary increments. They are symmetric α-stable stationary increments processes related to dissipative flows in the sense of Surgailis, Rosiński, Mandrekar and Cambanis (1998), and form an important subclass of all symmetric α-stable processes with stationary increments. They also can be viewed as stationary increments extensions of *stationary* mixed moving averages of Rosiński (1995).

To avoid trivialities, we will assume that

$$\operatorname{supp}\{G_t(x,u), t \in \mathbb{R}\} = X \times \mathbb{R} \quad \text{a.e. } \mu(dx)du \tag{1.4}$$

holds. By support $\operatorname{supp}\{G_t, t \in \mathbb{R}\}$ we mean a minimal (a.e.) set $A \subset X \times \mathbb{R}$ such that $m\{G_t(x,u) \neq 0, (x,u) \notin A\} = 0$ for every $t \in \mathbb{R}$, where $dm = \mu(dx)du$.

We shall focus on $S\alpha S$, $\alpha \in (0,2)$, mixed moving averages $\{X_\alpha(t)\}_{t\in\mathbb{R}}$ which are, in addition, *self-similar* with a self-similarity parameter $H > 0$, that is, for any $c > 0$,

$$\{X_\alpha(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{c^H X_\alpha(t)\}_{t\in\mathbb{R}}. \tag{1.5}$$

Such processes are of interest because their stationary increments can be used as models for strongly dependent $S\alpha S$ time series. Observe, however, that self-similarity imposes further restrictions on the kind of kernel functions G which can be used in the representation (1.1). Self-similar $S\alpha S$ mixed moving averages are studied in Pipiras and Taqqu (2002*a*, 2002*b*, 2003*a*, 2003*b*, 2003*c*). One can relate these processes to nonsingular flows. By using this relation and the structure of these flows, one can decompose self-similar mixed moving averages into distinct classes and then examine processes in each of the classes separately. The relation between self-similar mixed moving averages and flows established by Pipiras and Taqqu (2002*a*) (see Definition 5.1 and Proposition 5.1) is as follows.

Definition 1.1 (Pipiras and Taqqu (2002*a*)) A $S\alpha S$, $\alpha \in (0,2)$, self-similar process X_α having a mixed moving average representation (1.1) is said to be *generated by a nonsingular measurable flow* $\{\psi_c\}_{c>0}$ on $(X, \mathcal{X}, \mu)$ if, for all $c > 0$,

$$c^{-(H-1/\alpha)} G(x,cu) = b_c(x) \left\{ \frac{d(\mu\circ\psi_c)}{d\mu}(x) \right\}^{1/\alpha} G\Big(\psi_c(x), u + g_c(x)\Big) + j_c(x), \quad \text{a.e. } \mu(dx)du, \tag{1.6}$$

where $\{b_c\}_{c>0}$ is a cocycle for the flow $\{\psi_c\}_{c>0}$ taking values in $\{-1,1\}$, $\{g_c\}_{c>0}$ is a semi-additive functional for the flow $\{\psi_c\}_{c>0}$ and $j_c(x) : (0,\infty)\times X \to \mathbb{R}$ is some function, and also if (1.4) holds.

We shall now define the various terms used in Definition 1.1. A (multiplicative) *flow* $\{\psi_c\}_{c>0}$ is a collection of deterministic maps $\psi_c : X \to X$ such that

$$\psi_{c_1c_2}(x) = \psi_{c_1}(\psi_{c_2}(x)), \text{ for all } c_1, c_2 > 0, x \in X, \tag{1.7}$$

and $\psi_1(x) = x$, for all $x \in X$. A flow is *nonsingular* if $\mu(A) = 0$ implies $\mu(\psi_c^{-1}(A)) = 0$ for any $c > 0$ and $A \in \mathcal{X}$. It is *measurable* if the map $\psi_c(x) : (0,\infty) \times X \mapsto X$ is measurable. A *cocycle* $\{b_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ is a measurable map $b_c(x) : (0,\infty) \times X \mapsto Y$ satisfying the relation

$$b_{c_1 c_2}(x) = b_{c_1}(x) b_{c_2}(\psi_{c_1}(x)), \text{ for all } c_1, c_2 > 0, x \in X. \tag{1.8}$$

In our context, we will have either $Y = \{-1, 1\}$, $Y = (0,\infty)$ or $Y = \mathbb{R}\setminus\{0\}$. A *semi-additive functional* $\{g_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ is a measurable function $g_c(x) : (0,\infty) \times X \mapsto \mathbb{R}$ satisfying

$$g_{c_1 c_2}(x) = \frac{g_{c_1}(x)}{c_2} + g_{c_2}(\psi_{c_1}(x)), \text{ for all } c_1, c_2 > 0, x \in X. \tag{1.9}$$

All quantities entering in (1.6) thus obey a specific relation with the exception of the function $j_c(x)$. One of the goals of this work is to show that, without loss of generality, this function also obeys a special relation, namely,

$$j_{c_1 c_2}(x) = c_2^{-(H-1/\alpha)} j_{c_1}(x) + b_{c_1}(x) \left\{ \frac{d(\mu \circ \psi_{c_1})}{d\mu}(x) \right\}^{1/\alpha} j_{c_2}(\phi_{c_1}(x)), \text{ for all } c_1, c_2 > 0, x \in X. \tag{1.10}$$

It is quite easy to show that $\{j_c\}_{c>0}$ satisfies (1.10) a.e. $\mu(dx)$, for all $c_1, c_2 > 0$ (see the proof of Theorem 3.1), in which case $\{j_c\}_{c>0}$ is called *an almost 2-semi-additive functional*. Having (1.10) for a.e. $\mu(dx)$, however, is often not enough or difficult to solve for $\{j_c\}_{c>0}$ given a specific flow $\{\psi_c\}_{c>0}$. Therefore, we need to show that $\{j_c\}_{c>0}$ has a version (an a.e. modification for fixed c) satisfying (1.10) for all $x \in X$, which entails the specification of an adequate version of the Radon-Nikodym derivative $d(\mu \circ \psi_c)/d\mu$. A functional $\{j_c\}_{c>0}$ satisfying (1.10) will be called *2-semi-additive functional* (Definition 3.1).

Since g_c plays a role parallel to j_c, a functional $\{g_c\}_{c>0}$ satisfying (1.9) will be called *1-semi-additive functional*. *2*-semi-additive and other functionals are important when solving the equation (1.6) for the kernel function G related to some specific flows (see, for example, Pipiras and Taqqu (2003*c*), Theorem 3.1 and its proof). *2*-semi-additive functionals also shed light on the structure of our previous results (see the discussion following the proof of Theorem 3.1 below and the remark following Example 4.3 below).

The proof of the existence of a version which makes (1.10) valid for all $x \in X$, is involved. We establish this result in a more general context, namely, for the so-called *semi-additive functionals related to a cocycle*. After a simple transformation, *1*- and *2*-semi-additive functionals are examples of semi-additive functionals related to particular cocycles (see Examples 3.1 and 3.2).

We show not only the existence of a version but also obtain an expression for semi-additive functionals related to cocycles. We then use this expression to characterize *1*- and *2*-semi-additive functionals associated with cyclic flows.

The paper is organized as follows. Section 2 contains results on the semi-additive functionals related to cocycles. In Section 3, we prove that the function $j_c(x)$ in (1.6) can be taken as a *2*-semi-additive functional satisfying (1.10). We provide a number of examples in Section 4 which illustrate how one can derive the semi-additive functionals. Semi-additive functionals associated with cyclic flows are studied in Section 5.

2 Semi-additive functionals related to cocycles

For notational convenience, we will work here with additive flows and related functionals. Let $\{\phi_t\}_{t\in\mathbb{R}}$ be a measurable (additive) flow on a standard Lebesgue space $(X, \mathcal{X}, \mu)$ satisfying

$$\phi_{t_1+t_2}(x) = \phi_{t_1}(\phi_{t_2}(x))$$

for all $t_1, t_2 \in \mathbb{R}$, $x \in X$, and $\phi_0(x) = x$ for all $x \in X$. Let $\{A_t\}_{t\in\mathbb{R}}$ be a cocycle for the flow $\{\phi_t\}_{t\in\mathbb{R}}$ taking values in $\mathbb{R} \setminus \{0\}$, that is, a functional satisfying

$$A_{t_1+t_2}(x) = A_{t_1}(x)A_{t_2}(\phi_{t_1}(x)) \tag{2.1}$$

for all $t_1, t_2 \in \mathbb{R}$, $x \in X$. Observe that

$$A_0(x) = 1 \tag{2.2}$$

since (2.1) implies $A_0(x) = A_0(x)A_0(\phi_0(x)) = (A_0(x))^2$ by using $\phi_0(x) = x$.

Definition 2.1 A measurable map $F_t(x) : \mathbb{R} \times X \mapsto \mathbb{R}$ is called a *semi-additive functional related to a cocycle* $\{A_t\}_{t\in\mathbb{R}}$ (and a flow $\{\phi_t\}_{t\in\mathbb{R}}$) if, for all $t_1, t_2 \in \mathbb{R}$, $x \in X$,

$$F_{t_1+t_2}(x) = F_{t_1}(x) + A_{t_1}(x)F_{t_2}(\phi_{t_1}(x)). \tag{2.3}$$

When $F_t(x)$ satisfies (2.3) a.e. $\mu(dx)$, for all $t_1, t_2 \in \mathbb{R}$, it is called an *almost semi-additive functional related to a cocycle.*

Remark. In the context of multiplicative flows $\{\psi_c\}_{c>0}$, a semi-additive functional $\{J_c\}_{c>0}$ related to a cocycle $\{B_c\}_{c>0}$ is defined by the relation

$$J_{c_1c_2}(x) = J_{c_1}(x) + B_{c_1}(x)J_{c_2}(\psi_{c_1}(x)), \tag{2.4}$$

for all $c_1, c_2 > 0$, $x \in X$.

Notation. We write $\{\phi_t\}_{t\in\mathbb{R}}$, $\{A_t\}_{t\in\mathbb{R}}$ and $\{F_t\}_{t\in\mathbb{R}}$ in the context of additive flows and $\{\psi_c\}_{c>0}$, $\{B_c\}_{c>0}$ and $\{J_c\}_{c>0}$ in the context of multiplicative flows to denote respectively flows, cocycles and semi-additive functionals.

In the next example, we introduce a large class of semi-additive functionals related to cocycles.

Example 2.1 Let $\{A_t\}_{t\in\mathbb{R}}$ be a cocycle for the flow $\{\phi_t\}_{t\in\mathbb{R}}$ taking values in $\mathbb{R} \setminus \{0\}$. Let also $F : X \mapsto \mathbb{R}$ be a function and set

$$F_t(x) = A_t(x)F(\phi_t(x)) - F(x). \tag{2.5}$$

By using the cocycle equation (2.1), we have for $t_1, t_2 \in \mathbb{R}$,

$$\begin{aligned} F_{t_1+t_2}(x) &= A_{t_1}(x)A_{t_2}(\phi_{t_1}(x))F(\phi_{t_2}(\phi_{t_1}(x))) - F(x) \\ &= A_{t_1}(x)\Big(A_{t_2}(\phi_{t_1}(x))F(\phi_{t_2}(\phi_{t_1}(x))) - F(\phi_{t_1}(x))\Big) \\ &\qquad + A_{t_1}(x)F(\phi_{t_1}(x)) - F(x) \\ &= A_{t_1}(x)F_{t_2}(\phi_{t_1}(x)) + F_{t_1}(x). \end{aligned}$$

Therefore, in view of (2.3), the functional $\{F_t\}_{t\in\mathbb{R}}$ in (2.5) is a semi-additive functional related to the cocycle $\{A_t\}_{t\in\mathbb{R}}$.

Viewed from a different angle, (2.5) is a particular solution to the equation (2.3). It is not necessarily the unique solution. We will encounter in Section 4 a number of particular equations of the form (2.3), for which we derive the general solutions.

We now show that an almost semi-additive functional related to a cocycle has a version which is a semi-additive functional related to a cocycle. We say that $\{f_t\}_{t\in T} \subset L^0(S,\mathcal{S},m)$ is a *version* of $\{\widetilde{f}_t\}_{t\in T} \subset L^0(S,\mathcal{S},m)$, where T is an arbitrary index set, if $m(f_t \neq \widetilde{f}_t) = 0$ for all $t \in T$. The proof of this result uses the notion of a special flow $\widetilde{\phi}_t(y,u)$. Informally, the flow $\widetilde{\phi}_t(y,u)$ is defined on the set of points

$$\Omega = \{(y,u) : 0 \leq u < r(y), y \in Y\} = Y \times [0, r(\cdot)),$$

where $r(y)$ is a positive function. Plotting (y,u) in two dimensions, we can view the flow $\widetilde{\phi}_t$ as moving up vertically at constant speed till it reaches the level $r(y)$, and then jumps back to a point $(y',0)$ before it renews its vertical climb, this time from the point y'. Thus, if one focuses only on the horizontal Y axis, the flow starting at y moves to $y' = Vy$, then to $V^2y, \ldots, V^ny, \ldots$ Since the flow $\widetilde{\phi}_t$ moves constantly, observe that it has no fixed points.

We shall apply Theorem 3.1 of Kubo (1969) to define $\widetilde{\phi}_t$ formally. According to that theorem, any (measurable, nonsingular) flow $\{\phi_t\}_{t\in\mathbb{R}}$ without fixed points on a standard Lebesgue space is null-isomorphic (mod 0) to some *special flow* $\{\widetilde{\phi}_t\}_{t\in\mathbb{R}}$ defined on the space $(Y \times [0, r(\cdot)), \mathcal{Y} \otimes \mathcal{B}([0, r(\cdot))), \tau(dy)du)$ by

$$\widetilde{\phi}_t(y,u) = (V^n y, t + u - r_n(y)), \quad \text{for } 0 \leq u + t - r_n(y) < r(V^n y), \tag{2.6}$$

where $(Y, \mathcal{Y}, \tau)$ is a standard Lebesgue space, V is a null-isomorphism of Y onto itself, r is a positive measurable function on Y satisfying $\sum_{k=-\infty}^{-1} r(V^k y) = \sum_{k=0}^{\infty} r(V^k y) = \infty$, and where $r_n(y) = \sum_{k=0}^{n-1} r(V^k y)$ if $n \geq 1$, $r_n(y) = 0$ if $n = 0$, and $r_n(y) = \sum_{k=n}^{-1} r(V^k y)$ if $n \leq -1$. For additional intuition and information on special flows, see Chapter 11 in Cornfeld, Fomin and Sinai (1982), or Appendix A in Pipiras and Taqqu (2003*d*).

Theorem 2.1 *Suppose that $\{F_t\}_{t\in\mathbb{R}}$ is an almost semi-additive functional related to a cocycle $\{A_t\}_{t\in\mathbb{R}}$ and to a measurable, nonsingular flow $\{\phi_t\}_{t\in\mathbb{R}}$ on a standard Lebesgue space $(X, \mathcal{X}, \mu)$. Then, $\{F_t\}_{t\in\mathbb{R}}$ has a version which is a semi-additive functional related to the cocycle $\{A_t\}_{t\in\mathbb{R}}$.*

PROOF: We will extend the proof of Proposition 3.1 in Pipiras and Taqqu (2002*a*) and also use Proposition 1.2 in Kubo (1970). By using Remark 3.1 in Kubo (1969), it is enough to prove the proposition in the following two cases: *Case 1:* the flow $\{\phi_t\}_{t\in\mathbb{R}}$ is an identity flow, that is, $\phi_t(x) = x$ for all $t \in \mathbb{R}, x \in X$, and *Case 2:* the flow $\{\phi_t\}_{t\in\mathbb{R}}$ is a special flow as described above with the function r satisfying $r(y) \geq \theta$ for some fixed $\theta > 0$.

Case 1: If the flow $\{\phi_t\}_{t\in\mathbb{R}}$ is the identity, then the cocycle A_t in (2.1) must satisfy $A_t(x) = 1$ for all $t \in \mathbb{R}, x \in X$ (see, for example, Lemma 3.2 in Pipiras and Taqqu (2002*b*)). Relation (2.3) becomes

$$F_{t_1+t_2}(x) = F_{t_1}(x) + F_{t_2}(\phi_{t_1}(x)) \ \text{ a.e. } \mu(dx),$$

for all $t_1, t_2 \in \mathbb{R}$, which shows that the almost semi-additive functional $\{F_t\}_{t\in\mathbb{R}}$ is also an almost cocycle taking values in $\mathbb{R}$ but with addition as a group operation (it is of the form (2.1) but with a

sum instead of a product). Theorem B.9 in Zimmer (1984) implies that $\{F_t\}_{t\in\mathbb{R}}$ has a version which is a cocycle and hence, in our terminology, a semi-additive functional related to the cocycle $\{A_t\}_{t\in\mathbb{R}}$.

Case 2: Suppose that $\{\phi_t\}_{t\in\mathbb{R}}$ is a special flow on $Y\times[0,r(\cdot))$ as defined above and hence satisfies (2.6). For notational convenience, we shall write $F(t,(y,u))$ instead of $F_t(y,u)$. Since $F(t,\cdot)$ is a semi-additive functional related to a cocycle A_t, we have that, for any $s,t\in\mathbb{R}$,

$$F(s+t,(y,u)) = F(s,(y,u)) + A_s(y,u)\,F(t,(V^n y, u+s-r_n(y)))$$

a.e. for (y,u) such that $r_n(y)\le u+s<r_{n+1}(y)$. We can choose $u=u_0\in(0,\theta)$ and use the Fubini's theorem to conclude that

$$F(s+t,(y,u_0)) = F(s,(y,u_0)) + A_s(y,u_0)\,F(t,(V^n y, u_0+s-r_n(y))) \tag{2.7}$$

a.e. for (s,t,y) such that $r_n(y)\le u+s<r_{n+1}(y)$. Setting $n=0$ and $s+u_0=u$ in (2.7), we have

$$F(t,(y,u)) = (A_{u-u_0}(y,u_0))^{-1}F(t+u-u_0,(y,u_0)) - (A_{u-u_0}(y,u_0))^{-1}F(u-u_0,(y,u_0)) \tag{2.8}$$

a.e. for (t,y,u) such that $0\le u<r(y)$. We shall find expressions for the two F-terms on the right-hand side of (2.8). By using (2.7) with "s" and "t" indicated by the horizontal braces below, we have

$$F(t+u-u_0,(y,u_0)) = F(\underbrace{r_n(y)-v}_{s}+\underbrace{t+u-u_0-r_n(y)+v}_{t},(y,u_0)) = F(r_n(y)-v,(y,u_0))$$

$$+\,A_{r_n(y)-v}(y,u_0)F(t+u-u_0-r_n(y)+v,(V^m y, u_0+r_n(y)-v-r_m(y)))$$

a.e. for (t,y,u,v) such that

$$r_m(y)\le u_0+r_n(y)-v<r_{m+1}(y). \tag{2.9}$$

We can take $v=v_0\in(0,u_0)$ for which the relation above holds a.e. for (t,y,u) so that we have

$$0<v_0<u_0<\theta<r(y). \tag{2.10}$$

The inequality (2.9) implies that, for such v_0, we have $m=n$ and hence

$$F(t+u-u_0,(y,u_0)) = F(r_n(y)-v_0,(y,u_0))$$

$$+\,A_{r_n(y)-v_0}(y,u_0)F(t+u-u_0-r_n(y)+v_0,(V^n y,u_0-v_0)) \tag{2.11}$$

a.e. for (t,y,u). For the second term in (2.8), observe that, by (2.7) with $n=0$,

$$F(\underbrace{u-u_0}_{t}+\underbrace{v_0}_{s},(y,\underbrace{u_0-v_0}_{u_0})) = F(v_0,(y,u_0-v_0)) + A_{v_0}(y,u_0-v_0)\,F(u-u_0,(y,u_0))$$

and hence

$$F(u-u_0,(y,u_0)) = (A_{v_0}(y,u_0-v_0))^{-1}\Big(F(u-u_0+v_0,(y,u_0-v_0)) - F(v_0,(y,u_0-v_0))\Big) \tag{2.12}$$

a.e. for (y,u). Substituting (2.11) and (2.12) into (2.8), we obtain that

$$\begin{aligned} F(t,(y,u)) &= (A_{u-u_0}(y,u_0))^{-1}A_{r_n(y)-v_0}(y,u_0)F(t+u-u_0-r_n(y)+v_0,(V^n y,u_0-v_0)) \\ &\quad - (A_{u-u_0}(y,u_0))^{-1}(A_{v_0}(y,u_0-v_0))^{-1}F(u-u_0+v_0,(y,u_0-v_0)) \\ &\quad + (A_{u-u_0}(y,u_0))^{-1}F(r_n(y)-v_0,(y,u_0)) \\ &\quad + (A_{u-u_0}(y,u_0))^{-1}(A_{v_0}(y,u_0-v_0))^{-1}F(v_0,(y,u_0-v_0)) \qquad (2.13) \\ &=: F_1(t,(y,u))+F_2(t,(y,u)) \end{aligned}$$

a.e. (t,y,u), where F_1 and F_2 consist, respectively, of the first and last two terms in the sum (2.13). Since (2.11) holds for all $n\in\mathbb{Z}$, observe that the relation (2.13) does not depend on $n\in\mathbb{Z}$. We may therefore define $F_1(t,(y,u))$ and $F_2(t,(y,u))$ as above for $r_n(y)\le t+u<r_{n+1}(y)$, $n\in\mathbb{Z}$.

By using the cocycle equation (2.1),

$$\begin{aligned} &(A_{u-u_0}(y,u_0))^{-1}A_{r_n(y)-v_0}(y,u_0)A_{t+u-r_n(y)-u_0}(V^n y,u_0)A_{v_0}(V^n y,u_0-v_0) \\ &\qquad = (A_{u-u_0}(y,u_0))^{-1}A_{r_n(y)}(y,u_0)A_{t+u-r_n(y)-u_0}(V^n y,u_0) \\ &\qquad\quad = (A_{u-u_0}(y,u_0))^{-1}A_{t+u-u_0}(y,u_0)=A_t(y,u), \end{aligned}$$

since $(y,u)\in Y\times[0,r(\cdot))$. Then, we have

$$F_1(t,(y,u)) = A_t(y,u)\,F^*(\phi_t(y,u)) - F^*(y,u), \tag{2.14}$$

where

$$F^*(y,u)=(A_{u-u_0}(y,u_0))^{-1}(A_{v_0}(y,u_0-v_0))^{-1}F(u-u_0+v_0,(y,u_0-v_0)).$$

Example 2.1 shows that $F_1(t,\cdot)$, given by (2.14), is a semi-additive functional related to the cocycle A_t. We will now show that $F_2(t,\cdot)$ can be modified to a semi-additive functional related to the cocycle A_t.

It follows from the definition of the special flow that $\phi_t(y,u)=(y,t+u)$ for $0\le u+t<r(y)$, and thus $\phi_t(y,u)=(y,0)$ and $\phi_{v_0}(y,u_0-v_0)=(y,u_0)$ for all $0\le u_o<r(y)$. Using the cocycle relation (2.1), we get $A_{u-u_0}(y,u_0)=A_{-u_0}(y,u_0)A_u(y,0)$ and $A_{v_0-u_0}(y,u_0-v_0)=A_{v_0}(y,u_0-v_0)A_{-u_0}(y,u_0)$. Hence,

$$\begin{aligned} F_2(t,(y,u)) &= (A_u(y,0))^{-1}\Big\{(A_{-u_0}(y,u_0))^{-1}F(r_n(y)-v_0,(y,u_0)) \\ &\quad + (A_{v_0-u_0}(y,u_0-v_0))^{-1}F(v_0,(y,u_0-v_0))\Big\} =: (A_u(y,0))^{-1}G_n(y). \end{aligned} \tag{2.15}$$

By Lemma 2.1 below,

$$G_{n+m}(y)=G_n(y)+A_{r_n(y)}(y,0)\,G_m(V^n y) \quad \text{a.e. for } y. \tag{2.16}$$

It follows that

$$G_n(y)=\widetilde{G}_n(y) \text{ a.e. for } y,$$

where

$$\widetilde{G}_n(y)=\sum_{k\in[0,n)} A_{r_k(y)}(y,0)\,G_1(V^k y) \tag{2.17}$$

and $[0, n) = [n, 0)$ for $n < 0$. This can be checked by using

$$A_{r_{n+k}(y)}(y,0) = A_{r_n(y)}(y,0)A_{r_k(y)}(\phi_{r_n(y)}(y,0)) = A_{r_n(y)}(y,0)A_{r_k(y)}(V^n y,0)$$

to verify that $\widetilde{G}_n$ satisfies (2.16). By Lemma 2.2 below, the function

$$\widetilde{F}_2(t,(y,u)) = (A_u(y,0))^{-1}\widetilde{G}_n(y), \tag{2.18}$$

for $r_n(y) \leq u + t < r_{n+1}(y)$, is a semi-additive functional related to the cocycle A_t.

Since both F_1 and $\widetilde{F}_2$ are semi-additive functionals related to the cocycle A_t, so is the sum

$$\widetilde{F}(t,(y,u)) = F_1(t,(y,u)) + \widetilde{F}_2(t,(y,u))$$

and, since $F_2(t,(y,u)) = \widetilde{F}_2(t,(y,u))$ a.e. (t,y,u), we have

$$F(t,(y,u)) = \widetilde{F}(t,(y,u)) \tag{2.19}$$

a.e. (t,y,u).

To show that $\{\widetilde{F}(t,\cdot)\}_{t\in\mathbb{R}}$ is a version of $\{F(t,\cdot)\}_{t\in\mathbb{R}}$, it is enough to show that (2.19) holds also for all $t \in \mathbb{R}$, a.e. (y,u). The argument is standard. Set $\delta(t,(y,u)) = F(t,(y,u)) - \widetilde{F}(t,(y,u))$, $\Omega_t = \{(y,u) : \delta(t,(y,u)) = 0\}$ and also $\Omega_{s,t} = \{(y,u) : \delta(s+t,(y,u)) = \delta(s,(y,u)) + A_s(y,u)\delta(t,\phi_s(y,u))\}$. Denoting the Lebesgue measure on $\mathbb{R}$ by $\mathbb{L}$, we have $(\tau\otimes\mathbb{L})(\Omega_{s,t}^c) = 0$ for all $s,t \in \mathbb{R}$ but only $(\tau\otimes\mathbb{L})(\Omega_t^c) = 0$ a.e. for $t \in \mathbb{R}$. However, if $r \in \mathbb{R}$, then there are $s,t \in \mathbb{R}$ such that $s+t = r$ and $(\tau\otimes\mathbb{L})(\Omega_s^c) = (\tau\otimes\mathbb{L})(\Omega_t^c) = 0$. We also have $(\tau\otimes\mathbb{L})((\phi_{-s}\Omega_t)^c) = 0$ since ϕ_s is one-to-one and onto. Then, $(\tau\otimes\mathbb{L})((\Omega_{s,t}\cap\Omega_s\cap\phi_{-s}\Omega_t)^c) = 0$ and, for $(y,u) \in \Omega_{s,t}\cap\Omega_s\cap\phi_{-s}\Omega_t$, we have $\delta(r,(y,u)) = \delta(s,(y,u)) + A_s(y,u)\delta(t,\phi_s(y,u)) = 0$. This shows that $(\tau\otimes\mathbb{L})\{F(r,(y,u)) \neq \widetilde{F}(r,(y,u)\} = 0$ for any $r \in \mathbb{R}$, that is, $\{\widetilde{F}(t,\cdot)\}_{t\in\mathbb{R}}$ is a version of $\{F(t,\cdot)\}_{t\in\mathbb{R}}$. $\Box$

The theorem can be readily expressed in terms of multiplicative flows by changing time $t \in \mathbb{R}$ into $c = e^t$, $c > 0$.

Corollary 2.1 *Suppose that $\{J_c\}_{c>0}$ is an almost semi-additive functional related to a cocycle $\{B_c\}_{c>0}$ satisfying (2.4) and to a measurable, nonsingular multiplicative flow $\{\psi_c\}_{c>0}$ on a standard Lebesgue space $(X,\mathcal{X},\mu)$. Then, $\{J_c\}_{c>0}$ has a version which is a semi-additive functional related to the cocycle $\{B_c\}_{c>0}$.*

The next two corollaries, the first one for additive flows and the second for multiplicative flows, provide insight into the structure of semi-additive functionals related to a cocycle. Corollary 2.3 will be used in Propositions 5.1 and 5.2 below to deduce the forms of the *1*- and *2*-semi-additive functionals corresponding to a cyclic flow.

Corollary 2.2 *If the flow $\{\phi_t\}_{t\in\mathbb{R}}$ is given by its special representation (2.6) on a space $Y\times[0,r(\cdot))$, with the maps V and r, and $\{F_t\}_{t\in\mathbb{R}}$ is a semi-additive functional related to a cocycle $\{A_t\}_{t\in\mathbb{R}}$, then*

$$F_t(y,u) = F_t^{(1)}(y,u) + F_t^{(2)}(y,u), \tag{2.20}$$

where

$$
\begin{aligned}
F_t^{(1)}(y,u) &= A_t(y,u)\,F(\phi_t(y,u)) - F(y,u), &(2.21)\\
F_t^{(2)}(y,u) &= (A_u(y,0))^{-1}\sum_{k\in[0,n)} A_{r_k(y)}(y,0)F_1(V^k y), &(2.22)
\end{aligned}
$$

for $r_n(y)\le t+u<r_{n+1}(y)$, F, F_1 are some functions and $[0,n)=[n,0)$ for $n<0$. Moreover, each of the functions $F^{(1)}(t,\cdot)$ and $F^{(2)}(t,\cdot)$ is a semi-additive functional related to the cocycle $\{A_t\}_{t\in\mathbb{R}}$.

PROOF: The corollary follows from the proof of Theorem 2.1 by replacing "a.e." by "for all" conditions. See, in particular, (2.13), (2.14) and (2.15) together with (2.17). $\Box$

Corollary 2.2 involves an additive flow $\{\phi_t\}_{t\in\mathbb{R}}$. The next corollary formulates the result for a multiplicative flow $\{\psi_c\}_{c>0}$.

Corollary 2.3 *If $\{J_c\}_{c>0}$ is a semi-additive functional related to the cocycle $\{B_c\}_{c>0}$ and a* multiplicative *flow $\{\psi_c\}_{c>0}$, given by its special representation $\psi_c(y,u)=(V^n y, u+\ln c - r_n(y))$, then*

$$J_c(y,u) = J_c^{(1)}(y,u) + J_c^{(2)}(y,u), \tag{2.23}$$

where

$$
\begin{aligned}
J_c^{(1)}(y,u) &= B_c(y,u)\,J(\psi_c(y,u)) - J(y,u), &(2.24)\\
J_c^{(2)}(y,u) &= (B_{e^u}(y,0))^{-1}\sum_{k\in[0,n)} B_{e^{r_k(y)}}(y,0)J_1(V^k y), &(2.25)
\end{aligned}
$$

for $r_n(y)\le \ln c+u<r_{n+1}(y)$, and J, J_1 are some functions.

PROOF: This result follows by observing that $F_t(y,u):=J_{e^t}(y,u)$ is a semi-additive functional related to the cocycle $A_t(y,u)=B_{e^t}(y,u)$ and the additive flow $\phi_t(y,u)=\psi_{e^t}(y,u)$, applying Corollary 2.2 to $F_t(y,u)$ and then translating the result back in terms of the map $J_c(y,u)$. $\Box$

The following two auxiliary lemmas were used in the proof of Theorem 2.1.

Lemma 2.1 *Let $G_n(y)$ be defined by (2.15). Then, for $n,m\in\mathbb{Z}$,*

$$G_{n+m}(y) = G_n(y) + A_{r_n(y)}(y,0)G_m(V^n y) \ \text{a.e. } y. \tag{2.26}$$

PROOF: Recall from (2.15) that

$$G_n(y) = (A_{-u_0}(y,u_0))^{-1}F(r_n(y)-v_0,(y,u_0)) + (A_{v_0-u_0}(y,u_0-v_0))^{-1}F(v_0,(y,u_0-v_0)) \tag{2.27}$$

where u_0 and v_0 satisfy (2.10). To show (2.26), we shall use the relation

$$r_{n+m}(y) = r_n(y) + r_m(V^n y), \tag{2.28}$$

which is easy to verify by using the definition of $r_n(y)$. Observe that, for $n,m\in\mathbb{Z}$, by using (2.28) and (2.7),

$$F(r_{n+m}(y)-v_0,(y,u_0)) = F(\underbrace{r_n(y)-v_0}_{s}+\underbrace{r_m(V^n y)}_{t},(y,u_0))$$

$$
\begin{aligned}
&= F(r_n(y) - v_0, (y, u_0)) + A_{r_n(y)-v_0}(y, u_0)\, F(r_m(V^n y), (V^n y, u_0 - v_0)) \\
&= F(r_n(y) - v_0, (y, u_0)) + A_{r_n(y)-v_0}(y, u_0)\, F(\underbrace{v_0}_{s} + \underbrace{r_m(V^n y) - v_0}_{t}, (V^n y, u_0 - v_0)) \\
&= F(r_n(y) - v_0, (y, u_0)) + A_{r_n(y)-v_0}(y, u_0)\, F(v_0, (V^n y, u_0 - v_0)) \\
&\quad + A_{r_n(y)-v_0}(y, u_0) A_{v_0}(V^n y, u_0 - v_0)\, F(r_m(V^n y) - v_0, (V^n y, u_0)).
\end{aligned} \tag{2.29}
$$

To compute G_{n+m} we use (2.27) and substitute (2.29) for $F(r_{n+m}(y) - v_0, (y, u_0))$. This yields

$$
\begin{aligned}
G_{n+m}(y) &= (A_{-u_0}(y, u_0))^{-1} F(r_n(y) - v_0, (y, u_0)) + (A_{v_0-u_0}(y, u_0 - v_0))^{-1} F(v_0, (y, u_0 - v_0)) \\
&\quad + (A_{-u_0}(y, u_0))^{-1} A_{r_n(y)-v_0}(y, u_0) A_{v_0}(V^n y, u_0 - v_0)\, F(r_m(V^n y) - v_0, (V^n y, u_0)) \\
&\quad + (A_{-u_0}(y, u_0))^{-1} A_{r_n(y)-v_0}(y, u_0)\, F(v_0, (V^n y, u_0 - v_0)) \\
&= G_n(y) + (A_{-u_0}(y, u_0))^{-1} A_{r_n(y)-v_0}(y, u_0) A_{v_0}(V^n y, u_0 - v_0) A_{-u_0}(V^n y, u_0) \cdot \\
&\quad \cdot (A_{-u_0}(V^n y, u_0))^{-1} F(r_m(V^n y) - v_0, (V^n y, u_0)) \\
&\quad + (A_{-u_0}(y, u_0))^{-1} A_{r_n(y)-v_0}(y, u_0) A_{v_0-u_0}(V^n y, u_0 - v_0) \cdot \\
&\quad \cdot (A_{v_0-u_0}(V^n y, u_0 - v_0))^{-1} F(v_0, (V^n y, u_0 - v_0)) \\
&= G_n(y) + (A_{-u_0}(y, u_0))^{-1} A_{r_n(y)-v_0}(y, u_0) A_{v_0-u_0}(V^n y, u_0 - v_0)\, G_m(V^n y)
\end{aligned} \tag{2.30}
$$

a.e. y, where to obtain the last identity we used the relation $A_{v_0}(V^n y, u_0 - v_0) A_{-u_0}(V^n y, u_0) = A_{v_0-u_0}(V^n y, u_0 - v_0)$. Since $A_{-u_0}(y, u_0) A_{u_0}(y, 0) = A_{u_0-u_0}(y, 0) = A_0(y, 0) = 1$ (see (2.2)), we get $(A_{-u_0}(y, u_0))^{-1} = A_{u_0}(y, 0)$ and, since $A_{u_0}(y, 0) A_{r_n(y)-v_0}(y, u_0) A_{v_0-u_0}(V^n y, u_0 - v_0) = A_{r_n(y)-v_0+u_0}(y, 0) A_{v_0-u_0}(V^n y, u_0 - v_0) = A_{r_n(y)}(y, 0)$, we see that (2.30) reduces to (2.26). $\square$

Lemma 2.2 *If $F_t(y, u)$ is defined by the right-hand side of (2.18), together with (2.17), then $F_t(y, u)$ is a semi-additive functional related to the cocycle $\{A_t\}_{t \in \mathbb{R}}$.*

PROOF: Fix $s, t \in \mathbb{R}$ and $(y, u) \in Y \times [0, r(\cdot))$. We are interested in $F_s(y, u) = (A_u(y, 0))^{-1} \widetilde{G}_n(y)$ where n is such that $r_n(y) \le s + u < r_{n+1}(y)$ and in $F_{s+t}(y, u) = (A_u(y, 0))^{-1} \widetilde{G}_{n+m}(y)$ where, in addition, m is such that $r_{n+m}(y) \le t + s + u < r_{n+m+1}(y)$, an inequality equivalent to $r_m(V^n y) \le t + s + u - r_n(y) < r_{m+1}(V^n y)$ by (2.28). Then, by using the fact that $\widetilde{G}_n$ satisfies (2.26), we obtain that

$$
\begin{aligned}
A_s(y, u) F_t(\phi_s(y, u)) &= A_s(y, u) F_t(V^n y, s + u - r_n(y)) = A_s(y, u) (A_{s+u-r_n(y)}(V^n y, 0))^{-1} \widetilde{G}_m(V^n y) \\
&= A_s(y, u) (A_{s+u-r_n(y)}(V^n y, 0))^{-1} (A_{r_n(y)}(y, 0))^{-1} \left(\widetilde{G}_{n+m}(y) - \widetilde{G}_n(y) \right) \\
&= A_s(y, u) (A_{s+u}(y, 0))^{-1} \left(\widetilde{G}_{n+m}(y) - \widetilde{G}_n(y) \right) = (A_u(y, 0))^{-1} \left(\widetilde{G}_{n+m}(y) - \widetilde{G}_n(y) \right) \\
&= F_{s+t}(y, u) - F_s(y, u).
\end{aligned}
$$

This concludes the proof. $\square$

3 $\mathcal{2}$-semi-additive functionals

In this section, we apply Corollary 2.1 to show that the function $j_c(x)$ in the relation (1.6) can be chosen, without loss of generality, as a $\mathcal{2}$-semi-additive functional satisfying (1.10). Observe first that because of the properties of the Radon-Nikodym derivatives, one has for all $c_1, c_2 > 0$,

$$\begin{aligned} \frac{d(\mu \circ \psi_{c_1 c_2})}{d\mu}(x) &= \frac{d(\mu \circ \psi_{c_1})}{d\mu}(x) \frac{d(\mu \circ \psi_{c_2} \circ \psi_{c_1})}{d(\mu \circ \psi_{c_1})}(x) \\ &= \frac{d(\mu \circ \psi_{c_1})}{d\mu}(x) \frac{d(\mu \circ \psi_{c_2})}{d\mu}(\psi_{c_1}(x)) \end{aligned} \tag{3.1}$$

a.e. $\mu(dx)$, which is the relation (1.8) defining a cocycle but valid only a.e. $\mu(dx)$ and not for all $x \in X$. We start with the following lemma which shows that the collection of the Radon-Nikodym derivatives $\{d(\mu \circ \psi_c)/d\mu\}_{c>0}$ has a version which is a cocycle for all $x \in X$.

Lemma 3.1 *Suppose that the relations (1.6) and (1.4) hold. Then, the Radon-Nikodym derivatives* $\{d(\mu \circ \psi_c)/d\mu\}_{c>0}$ *have a version which is*
1) jointly measurable in (c, x)*,*
2) a cocycle mapping $(0, \infty) \times X \to (0, \infty)$*,*
3) a Radon-Nikodym derivative $d(\mu \circ \psi_c)/d\mu$ *for all* $c > 0$*.*

PROOF: We first show that the collection $\{d(\mu \circ \psi_c)/d\mu\}_{c>0}$ has a jointly measurable version. By using the notation (1.3), relation (1.6) implies that, for any $t \in \mathbb{R}$ and $c > 0$,

$$G_{ct}(x, cu) = c^{H-1/\alpha} b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha} G_t(\psi_c(x), u + g_c(x)) \tag{3.2}$$

a.e. $\mu(dx)du$.

If $\mathrm{supp}\{G_t(x, u)\} = X \times \mathbb{R}$ a.e. $\mu(dx)du$ for some fixed $t \in \mathbb{R}$, we have for $c > 0$,

$$\frac{d(\mu \circ \psi_c)}{d\mu}(x) = \left\{ \frac{c^{1/\alpha - H} G_{ct}(x, cu)}{b_c(x) G_t(\psi_c(x), u + g_c(x))} \right\}^{\alpha}$$

a.e. $\mu(dx)du$. Hence, since the right-hand side of the expression above is jointly measurable, we may conclude that $\{d(\mu \circ \psi_c)/d\mu\}_{c>0}$ has a jointly measurable version.

Consider now the general case when $\mathrm{supp}\{G_t(x, u)\} = X \times \mathbb{R}$ a.e. $\mu(dx)du$ may possibly not hold for any $t \in \mathbb{R}$. Let $\mathcal{G} = \mathrm{Sp}\{G_t, t \in \mathbb{R}\}$ be the linear span of G_t, $t \in \mathbb{R}$, and $\overline{\mathcal{G}}$ be the closure of $\mathcal{G}$ in the space $L^\alpha(X \times \mathbb{R}, \mu(dx)du)$. Since $\{G_t, t \in \mathbb{R}\} \subset \overline{\mathcal{G}}$, the assumption (1.4) implies that $\mathrm{supp}\{\overline{\mathcal{G}}\} = X \times \mathbb{R}$ a.e. $\mu(dx)du$. By Lemma 3.2 in Hardin (1981), there is a function $G^* \in \overline{\mathcal{G}}$ such that $\mathrm{supp}\{G^*(x, u)\} = X \times \mathbb{R}$ a.e. $\mu(dx)du$. Since $G^* \in \overline{\mathcal{G}}$, there are functions $G^{(n)}(x, u) = \sum_i a_{ni} G_{t_{ni}}(x, u) \in \mathcal{G}$, $n \geq 1$, $a_{ni}, t_{ni} \in \mathbb{R}$, such that $G^{(n)}(x, u) \to G^*(x, u)$ a.e. $\mu(dx)du$.

Let also $G_c^{(n)}(x, u) = \sum_i a_{ni} G_{ct_{ni}}(x, cu)$. Relation (3.2) implies that, for any $c > 0$ and $n \geq 1$,

$$G_c^{(n)}(x, u) = c^{H-1/\alpha} b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha} G^{(n)}(\psi_c(x), u + g_c(x)) \tag{3.3}$$

a.e. $\mu(dx)du$. For any $c>0$, the right-hand side of (3.3) converges to

$$c^{H-1/\alpha} b_c(x) \left\{ \frac{d(\mu\circ\psi_c)}{d\mu}(x) \right\}^{1/\alpha} G^*(\psi_c(x), u + g_c(x))$$

a.e. $\mu(dx)du$, as $n\to\infty$. Since the right-hand side of (3.3) converges, the left-hand side of (3.3) converges to some function $G_c^*(x,u)$. Hence, for any $c>0$,

$$G_c^*(x,u) = c^{H-1/\alpha} b_c(x) \left\{ \frac{d(\mu\circ\psi_c)}{d\mu}(x) \right\}^{1/\alpha} G^*(\psi_c(x), u + g_c(x))$$

or, since $\mathrm{supp}\{G^*\} = X\times\mathbb{R}$ a.e.,

$$\frac{d(\mu\circ\psi_c)}{d\mu}(x) = \left\{ \frac{c^{1/\alpha - H} G_c^*(x,u)}{b_c(x) G^*(\psi_c(x), u+g_c(x))} \right\}^{\alpha} \tag{3.4}$$

a.e. $\mu(dx)du$. Observe that $G_c^*(x,u)$ is jointly measurable in (c,x,u) because it is the a.e. limit of functions jointly measurable in (c,x,u). Since $G^*(x,u)$ is measurable in (x,u), $\psi_c(x), g_c(x)$ and $b_c(x)$ are measurable in (c,x), the function $c^{H-1/\alpha} b_c(x)^{1/\alpha} G^*(\psi_c(x), u+g_c(x))$ is measurable in (c,x,u). Hence, the right-hand side of (3.4) is jointly measurable which is to say that $\{d(\mu\circ\psi_c)/d\mu\}_{c>0}$ has a jointly measurable version.

Suppose then, without loss of generality, that $d(\mu\circ\psi_c)/d\mu(x)$ is jointly measurable. We still need to show that $\{d(\mu\circ\psi_c)/d\mu\}_{c>0}$ has a version which is a cocycle. Since the flow $\{\psi_c\}_{c>0}$ is nonsingular, the measures $\mu\circ\psi_c$ and μ are equivalent and hence we may suppose that $(d(\mu\circ\psi_c)/d\mu)(x) : (0,\infty)\times X \to \mathbb{R}\setminus\{0\}$. By (3.1), $\{d(\mu\circ\psi_c)/d\mu\}_{c>0}$ is an *almost cocycle* for the flow $\{\psi_c\}_{c>0}$ where "almost" refers to the fact that the relation (3.1) holds a.e. $\mu(dx)$ for $c_1, c_2 > 0$, in contrast to (1.8) which holds for all $x\in X$ and $c_1,c_2>0$. By Theorem B.9 in Zimmer (1984) (see also Theorem A.1 in Kolodyński and Rosiński (2002)) and since $(d(\mu\circ\psi_c)/d\mu)(x)$ is measurable in (c,x), $\{d(\mu\circ\psi_c)/d\mu\}_{c>0}$ has a version which is a cocycle for the flow $\{\psi_c\}_{c>0}$ taking values in $(0,\infty)$. Property 3) follows from the definition of "version". □

Remark. The version specified in Lemma 3.1 which satisfies Conditions 1), 2) and 3) is not unique. Suppose, for instance, that $X = \mathbb{R}^2 = \{(x_1,x_2)\}$, $\mu(dx) = dx_1dx_2$ and $\psi_c(x_1,x_2) = (x_1, x_2 + \ln c)$. Then,

$$\frac{d(\mu\circ\psi_c)}{d\mu}(x_1,x_2) \equiv 1$$

is a version of the Radon-Nikodym derivatives satisfying Conditions 1), 2) and 3). On the other hand, let $b:\mathbb{R}\mapsto(0,\infty)$ be an arbitrary function and $x_1^*\in\mathbb{R}$ be fixed. Then,

$$\frac{d(\mu\circ\psi_c)}{d\mu}(x_1,x_2) = \begin{cases} 1, & x_1 \neq x_1^*, \\ \frac{b(x_2+\ln c)}{b(x_2)}, & x_1 = x_1^*, \end{cases}$$

is also a version of the Radon-Nikodym derivatives satisfying Conditions 1), 2) and 3). Indeed, it is jointly measurable and also, for fixed $c>0$, it is still a Radon-Nikodym derivative since it was modified on the set $\{(x_1,x_2) : x_1 = x_1^*\}$ of a μ-measure zero. It satisfies a cocycle equation for all $x\in\mathbb{R}^2$,

$c_1, c_2 > 0$, because it does so on the disjoint subsets $\{(x_1, x_2) : x_1 \neq x_1^*\}$ and $\{(x_1, x_2) : x_1 = x_1^*\}$ of $\mathbb{R}^2$ which are invariant under the flow. Observe that the two versions of the Radon-Nikodym derivatives above are different when $b \not\equiv 1$.

We can now give a precise definition of *2*-semi-additive functional.

Definition 3.1 A measurable function $j_c(x) : (0, \infty) \times X \to \mathbb{R}$ is a *2-semi-additive functional* for a flow $\{\psi_c\}_{c>0}$ and a cocycle $\{b_c\}_{c>0}$ if the relation (1.10) holds, where $d(\mu \circ \psi_c)/d\mu$ satisfies Conditions 1), 2) and 3) of Lemma 3.1. A semi-additive functional $\{g_c\}_{c>0}$ satisfying (1.9) is called a *1-semi-additive functional.*

In the following examples, we show that after multiplication by a suitable factor, the *1*-semi-additive functional $\{g_c\}_{c>0}$ in (1.9) and the *2*-semi-additive functionals $\{j_c\}_{c>0}$ in (1.10) become semi-additive functionals related to a cocycle, and we identify these cocycles.

Example 3.1 If $\{g_c\}_{c>0}$ is a *1*-semi-additive functional satisfying (1.9), then $J_c(x) = cg_c(x)$ satisfies

$$J_{c_1 c_2}(x) = J_{c_1}(x) + c_1 J_{c_2}(\psi_{c_1}(x)), \quad \text{for all } c_1, c_2 > 0, x \in X.$$

But $B_c(x) = c$ is a cocycle for the flow $\{\psi_c\}_{c>0}$ (it satisfies (1.8)). Therefore, $\{J_c\}_{c>0}$ is a semi-additive functional related to the cocycle $\{B_c\}_{c>0}$.

Example 3.2 If $\{j_c\}_{c>0}$ is a *2*-semi-additive functional satisfying (1.10), then $J_c(x) = c^{H-1/\alpha} j_c(x)$ satisfies

$$J_{c_1 c_2}(x) = J_{c_1}(x) + B_{c_1}(x) J_{c_2}(\psi_{c_1}(x)), \quad \text{for all } c_1, c_2 > 0, x \in X$$

with

$$B_c(x) = c^{H-1/\alpha} b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha}. \tag{3.5}$$

Since $\{b_c\}_{c>0}$ is a cocycle taking values in $\{-1, 1\}$ and $\{d(\mu \circ \psi_c)/d\mu\}_{c>0}$ is a cocycle taking values in $\mathbb{R} \setminus \{0\}$, it is easy to check that $\{B_c\}_{c>0}$ is also a cocycle taking values in $\mathbb{R} \setminus \{0\}$. Thus, $\{J_c\}_{c>0}$ is a semi-additive functional related to the cocycle (3.5).

Remarks

1. The cocycles $\{B_c\}_{c>0}$ appearing in the preceding examples are associated with functionals $\{J_c\}_{c>0}$. They should not be confused with the cocycle $\{b_c\}_{c>0}$ in Relation (1.6).
2. The preceding examples can be used in the following way. Suppose that $\{g_c\}_{c>0}$ and $\{j_c\}_{c>0}$ are only *almost* semi-additive functionals. Then $\{J_c\}_{c>0}$ would also be almost semi-additive functionals. Since Corollary 2.3 applies to $\{J_c\}_{c>0}$, these have a version which is a semi-additive functional. In view of the expressions relating J_c to g_c and j_c, it follows that $\{g_c\}_{c>0}$ and $\{j_c\}$ have also a version which is a semi-additive functional. This type of argument is used in the proof of the following theorem.

Theorem 3.1 *Let $\alpha \in (0, 2)$ and $H > 0$. The function $j_c(x)$ in relation (1.6) can be taken to be a 2-semi-additive functional.*

PROOF: We need first to show that the function $j_c(x)$ in (1.6) is an almost semi-additive function. Observe that it equals

$$j_c(x) = c^{-(H-1/\alpha)} G(x, cu) - \widetilde{b}_c(x)\, G(\psi_c(x), u + g_c(x)), \tag{3.6}$$

a.e. $\mu(dx)du$, for any $c > 0$, where

$$\widetilde{b}_c(x) = b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha}. \tag{3.7}$$

By Lemma 3.1 above, the Radon-Nikodym derivative $d(\mu \circ \psi_c)/d\mu$ and hence its $1/\alpha$-power have a version which is a cocycle taking values in $(0, \infty)$. Since $b_c(x)$ is a cocycle, the product $\widetilde{b}_c(x)$ also has a version which is a cocycle taking values in $\mathbb{R} \setminus \{0\}$. We may therefore suppose without loss of generality that $\{\widetilde{b}_c\}_{c>0}$ is a cocycle in (3.6), that is,

$$\widetilde{b}_{c_1 c_2}(x) = \widetilde{b}_{c_1}(x) \widetilde{b}_{c_2}(\psi_{c_1}(x)). \tag{3.8}$$

By using (3.6), we have for $c_1, c_2 > 0$,

$$j_{c_1 c_2}(x) = (c_1 c_2)^{-(H-1/\alpha)} G(x, c_1 c_2 u) - \widetilde{b}_{c_1 c_2}(x)\, G(\psi_{c_1 c_2}(x), u + g_{c_1 c_2}(x))$$

a.e. $\mu(dx)du$. By using (3.8) and $g_{c_1 c_2}(x) = c_2^{-1} g_{c_1}(x) + g_{c_2}(\psi_{c_1}(x))$, we conclude that

$$\begin{aligned} j_{c_1 c_2}(x) &= c_2^{-(H-1/\alpha)} \Big(c_1^{-(H-1/\alpha)} G(x, c_1(c_2 u)) - \widetilde{b}_{c_1}(x)\, G(\psi_{c_1}(x), c_2 u + g_{c_1}(x)) \Big) \\ &\qquad + \widetilde{b}_{c_1}(x) \Big\{ c_2^{-(H-1/\alpha)} G\Big(\psi_{c_1}(x), c_2(u + c_2^{-1} g_{c_1}(x))\Big) \\ &\qquad\qquad - \widetilde{b}_{c_2}(\psi_{c_1}(x)) G\Big(\psi_{c_2}(\psi_{c_1}(x)), u + c_2^{-1} g_{c_1}(x) + g_{c_2}(\psi_{c_1}(x))\Big) \Big\} \\ &= c_2^{-(H-1/\alpha)} j_{c_1}(x) + \widetilde{b}_{c_1}(x) j_{c_2}(\psi_{c_1}(x)) \end{aligned} \tag{3.9}$$

a.e. $\mu(dx)$. Hence, $\{j_c\}_{c>0}$ is an almost semi-additive functional.

Multiplying (3.9) by $(c_1 c_2)^{H-1/\alpha}$ and setting $J_c(x) = c^{H-1/\alpha} j_c(x)$, $B_c(x) = c^{H-1/\alpha} \widetilde{b}_c(x)$, we obtain that

$$J_{c_1, c_2}(x) = J_{c_1}(x) + B_{c_1}(x) J_{c_2}(\psi_{c_1}(x)) \quad \text{a.e. } \mu(dx). \tag{3.10}$$

Since $\{B_c\}_{c>0}$ is also a cocycle for the flow $\{\psi_c\}_{c>0}$, relation (3.10) shows that $\{J_c\}_{c>0}$ is an almost semi-additive functional related to the cocycle $\{B_c\}_{c>0}$ in the sense of Definition 2.1 below. By Corollary 2.1, $\{J_c\}_{c>0}$ has a version which is a semi-additive functional related to the cocycle $\{B_c\}_{c>0}$. But $j_c(x) = c^{-(H-1/\alpha)} J_c(x)$. Hence, when multiplied by $c^{-(H-1/\alpha)}$, this version is a *2*-semi-additive functional which is a version of $\{j_c\}_{c>0}$. $\Box$

The corresponding result for g_c was proved in Proposition 3.1 of Pipiras and Taqqu (2002*a*), pages 421-426. (In that proposition, "semi-additive functional" means "*1*-semi-additive-functional".) The proof of that Proposition 3.1 also follows from the more general setting of the present paper. In fact, it reduces, at this stage, to the argument in Example 3.1 and the remark preceding the statement of Theorem 3.1.

Theorem 3.1 is useful when solving for the kernel function G generated by a given flow. One such case, studied in Proposition 3.1 of Pipiras and Taqqu (2003*c*), concerns kernel functions related to cyclic flows. *2*-semi-additive functionals could have also been used in Theorem 5.1 of Pipiras and Taqqu (2002*b*), where kernels related to identity flows of Example 4.1 below are considered. In particular, the argument used in Theorem 5.1 of that paper involving an almost everywhere version of the Cauchy functional equation would not be necessary anymore. We explain this in greater detail in the remark following Example 4.3 below.

4 Examples

In the following examples, we consider *1*- and *2*-semi-additive functionals $\{j_c\}_{c>0}$ for identity, dissipative and cyclic flows $\{\psi_c\}_{c>0}$, and related cocycles $\{b_c\}_{c>0}$.

Example 4.1 Consider the *identity* flow $\{\psi_c\}_{c>0}$ on (X, μ) such that

$$\psi_c(x) = x \tag{4.1}$$

for all $c > 0$ and $x \in X$. We can take

$$\frac{d(\mu \circ \psi_c)}{d\mu}(x) = \frac{d\mu}{d\mu}(x) \equiv 1 \tag{4.2}$$

as a cocycle for the identity flow $\{\psi_c\}_{c>0}$. By Lemma 3.2 in Pipiras and Taqqu (2002*b*),

$$b_c(x) = 1 \tag{4.3}$$

for the identity flow $\{\psi_c\}_{c>0}$. The *2*-semi-additive functional $\{j_c\}_{c>0}$ in (1.10) therefore satisfies

$$j_{c_1c_2}(x) = c_2^{-(H-1/\alpha)} j_{c_1}(x) + j_{c_2}(x), \tag{4.4}$$

for all $x \in X$, $c_1, c_2 > 0$. Relation (4.4) is an equation for the functional $j_c(x)$, which we shall now solve.

If $H \neq 1/\alpha$, by subtracting $j_{c_1c_2}(x) = c_1^{-(H-1/\alpha)} j_{c_2}(x) + j_{c_1}(x)$ from (4.4), we obtain that

$$(1 - c_2^{-(H-1/\alpha)}) j_{c_1}(x) = (1 - c_1^{-(H-1/\alpha)}) j_{c_2}(x).$$

By fixing $c_2 \neq 1$, we conclude that

$$j_c(x) = j(x)(1 - c^{-(H-1/\alpha)}),$$

where $j(x)$ is some function. If $H = 1/\alpha$, then

$$j_{c_1c_2}(x) = j_{c_2}(x) + j_{c_1}(x)$$

and by using Lemma 1.1.6 in Bingham et al. (1987), we have

$$j_c(x) = j(x) \ln c,$$

where $j(x)$ is some function.

Example 4.2 Consider the flow

$$\psi_c(y,u) = (y, u + \ln c) \tag{4.5}$$

on the space $(X,\mu) = (Y \times \mathbb{R}, \nu(dy)du)$. By Krengel's theorem (see, for example, Theorem 3.1 in Pipiras and Taqqu (2002*b*)), any *dissipative* flow on (X,μ) is null-isomorphic to a flow $\{\psi_c\}_{c>0}$ of the form (4.5). Let $\mathbb{L}$ denote the Lebesgue measure on $\mathbb{R}$. We can take

$$\frac{d((\nu\otimes\mathbb{L})\circ\psi_c)}{d(\nu\otimes\mathbb{L})}(y,u) = \frac{d\nu(y)}{d\nu(y)}\frac{d(u+\ln c)}{du} \equiv 1$$

as a cocycle for the flow $\{\psi_c\}_{c>0}$. By Lemma 3.1 in Pipiras and Taqqu (2002*b*), for the dissipative flow $\{\psi_c\}_{c>0}$, the cocycle b_c is given by

$$b_c(y,u) = \frac{b(\psi_c(y,u))}{b(y,u)} \tag{4.6}$$

with some function b taking values in $\{-1,1\}$. Hence, a *2*-semi-additive functional $\{j_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ in (1.10) satisfies

$$j_{c_1c_2}(y,u) = c_2^{-(H-1/\alpha)} j_{c_1}(y,u) + \frac{b(\psi_c(y,u))}{b(y,u)} j_{c_2}(\psi_{c_1}(y,u))$$

for all $(y,u) \in Y \times \mathbb{R}$ and $c_1, c_2 > 0$.

To solve this equation for j_c, set $\widetilde{j}_c(y,u) = b(y,u)j_c(y,u)$ so that

$$\widetilde{j}_{c_1c_2}(y,u) = c_2^{-(H-1/\alpha)}\widetilde{j}_{c_1}(y,u) + \widetilde{j}_{c_2}(y, u + \ln c_1).$$

Substituting $u=0$ into this relation and setting $\ln c_1 = v$ so that $c_1 = e^v$, $c_2 = c$ and $\widetilde{j}(y,s) = \widetilde{j}_{e^s}(y,0)$, we obtain that $c_1c_2 = e^{v+\ln c}$ and

$$\widetilde{j}(y, v+\ln c) = c^{-(H-1/\alpha)}\widetilde{j}(y,v) + \widetilde{j}_c(y,v).$$

Hence,

$$\begin{aligned} j_c(y,u) = (b(y,u))^{-1}\widetilde{j}_c(y,u) &= \frac{\widetilde{j}(\psi_c(y,u))}{b(y,u)} - c^{-(H-1/\alpha)}\frac{\widetilde{j}(y,u)}{b(y,u)} \\ &= b_c(y,u)j(\psi_c(y,u)) - c^{-(H-1/\alpha)}j(y,u), \end{aligned}$$

by (4.6), where $j(y,u) = \widetilde{j}(y,u)/b(y,u)$ is some function.

Example 4.3 In view of (1.9), a *1*-semi-additive functional $\{g_c\}_{c>0}$ for the identity flow (4.1) satisfies $g_{c_1c_2}(x) = c_2^{-1}g_{c_1}(x) + g_{c_2}(x)$ whose solution is

$$g_c(x) = (c^{-1} - 1)g(x)$$

for some function $g : X \mapsto \mathbb{R}$ (Lemma 3.2 in Pipiras and Taqqu (2002*b*)). The corresponding equation for the dissipative flow (4.5) is $g_{c_1c_2}(y,u) = c_2^{-1}g_{c_1}(y,u) + g_{c_2}(y, u + \ln c_2)$ where solution is

$$g_c(y,u) = g(y, u+\ln c) - c^{-1}g(y,u),$$

for some function $g : Y \times \mathbb{R} \mapsto \mathbb{R}$ (Lemma 3.2 in Pipiras and Taqqu (2002*b*)).

Remark. Consider Relation (1.6) where the flow $\{\psi_c\}_{c>0}$ is the identity. Observe that, by using the relations (4.2) and (4.3) and Examples 4.1 and 4.3, the equation (1.6) becomes: for any $c > 0$,

$$c^{-(H-1/\alpha)}G(x, cu) = G\Big(x, u + (c^{-1} - 1)g(x)\Big) + (1 - c^{-(H-1/\alpha)})j(x)$$

a.e. $\mu(dx)du$, when $H \neq 1/\alpha$, and

$$G(x, cu) = G\Big(x, u + (c^{-1} - 1)g(x)\Big) + (\ln c)j(x)$$

a.e. $\mu(dx)du$, when $H = 1/\alpha$. These equations were also obtained in the proof of Theorem 5.1 in Pipiras and Taqqu (2002*b*) and then used to solve for the function G. The arguments in that theorem, leading to the two equations were quite involved but as we see here, the equations follow easily once *2*-semi-additive functionals are used.

Example 4.4 For $v \in \mathbb{R}$ and $a > 0$, let

$$[v]_a = \max\{n \in \mathbb{Z} : na \leq v\}, \quad \{v\}_a = v - a[v]_a. \tag{4.7}$$

By Theorem 2.1 in Pipiras and Taqqu (2003*d*), any *cyclic flow* is null-isomorphic to the flow

$$\psi_c(z, v) = (z, \{v + \ln c\}_{q(z)}) \tag{4.8}$$

on the space $(X, \mu) = (Z \times [0, q(\cdot)), \sigma(dz)dv)$, where $q(z) > 0$ is some function. Observe that $\{v + \ln c\}_{q(z)}$, as a function of v, has the shape of a seesaw. Relation (1.9) for the *1*-semi-additive functional $\{g_c\}_{c>0}$ of the flow (4.8) becomes $g_{c_1c_2}(z, v) = c_2^{-1}g_{c_1}(z, v) + g_{c_2}(z, \{v + \ln c_2\}_{q(z)})$. The solution to this equation, which is given in Proposition 5.1 below, is as follows:

$$g_c(z, v) = g(z, \{v + \ln c\}_{q(z)}) - c^{-1}g(z, v), \tag{4.9}$$

for some function $g : Z \times [0, q(\cdot)) \mapsto \mathbb{R}$.

Example 4.5 We now consider *2*-semi-additive functionals for cyclic flows (4.8). By Lemma 8.2 in Pipiras and Taqqu (2003*c*), the cocycle $b_c(z, v)$ in (1.6) for the flow $\{\psi_c\}_{c>0}$ can be expressed as

$$b_c(z, v) = b_1(z)^{[v+\ln c]_{q(z)}} \frac{b(\psi_c(z, v))}{b(z, v)} \tag{4.10}$$

for some functions $b_1 : Z \mapsto \{-1, 1\}$ and $b : Z \times [0, q(\cdot)) \mapsto \{-1, 1\}$. The Radon-Nikodym derivatives

$$\frac{d((\sigma \otimes \mathbb{L}) \circ \psi_c)}{d(\sigma \otimes \mathbb{L})}(z, v) \equiv 1$$

because $d\{v + \ln c_2\}_{q(z)}/dv = 1$ for almost all v since $q(z)$ does not affect the slope. These Radon-Nikodym derivatives can be taken as a cocycle for the flow $\{\psi_c\}_{c>0}$. The *2*-semi-additive functional $\{j_c\}_{c>0}$ in (1.10) therefore satisfies

$$j_{c_1c_2}(z, v) = c_2^{-(H-1/\alpha)} j_{c_1}(z, v) + b_1(z)^{[v+\ln c]_{q(z)}} \frac{b(\psi_c(z, v))}{b(z, v)} j_{c_2}(z, v).$$

The solution to this equation, which is given in Proposition 5.2 below, is as follows:

$$j_c(z,v) = b_1(z)^{[v+\ln c]_{q(z)}} \frac{b(\psi_c(z,v))}{b(z,v)} j(\psi_c(z,v)) - c^{-(H-1/\alpha)} j(z,v)$$
$$+ \frac{j_1(z)}{b(z,v)}[v+\ln c]_{q(z)} 1_{\{b_1(z)=1\}} 1_{\{H=1/\alpha\}}, \qquad (4.11)$$

for some functions $j_1 : Z \mapsto \mathbb{R}$ and $j : Z \times [0, q(\cdot)) \mapsto \mathbb{R}$.

5 Semi-additive functionals for cyclic flows

In this section, we solve the *1*- and *2*-semi-additive functional equations (1.9) and (1.10) for the cyclic flows $\{\psi_c\}_{c>0}$ of the form (4.8). The results are used in Pipiras and Taqqu (2003c) to obtain a general form for the kernel G of a mixed moving average generated by a cyclic flow.

By (4.7), one has $\psi_c(z,v) = (z, \{\ln c + v\}_{q(z)}) = (z, \ln c + v - nq(z))$ when $nq(z) \le \ln c + v < (n+1)q(z)$ and hence these flows have the special representation (2.6) with t replaced by $\ln c$ and

$$V(z) = z, \quad r_n(z) = nq(z). \qquad (5.1)$$

This representation is convenient when applying Corollary 2.3 to characterize semi-additive functionals as in the following propositions.

Proposition 5.1 *Let $\{\psi_c\}_{c>0}$ be a cyclic flow on the space $Z \times [0, q(\cdot))$ given by (4.8), and let $\{g_c\}_{c>0}$ be a 1-semi-additive functional for the flow $\{\psi_c\}_{c>0}$ satisfying (1.9). Then, the solution to the equation (1.9) is given by (4.9).*

PROOF: Example 3.1 shows that $J_c(z,v) = cg_c(z,v)$ is a semi-additive functional in the sense of (2.4) for the cyclic flow $\{\psi_c\}_{c>0}$ and the cocycle $\{B_c\}_{c>0}$ defined by $B_c(z,v) = c$. Since $\psi_c(z,v)$ has a special representation (2.6) with t replaced by $\ln c$, and V and r_n defined in (5.1), Corollary 2.3 shows that the semi-additive functional $\{J_c\}_{c>0}$ can be expressed as the sum of two semi-additive functionals. After substituting $J_c(z,v) = cg_c(z,v)$ into their expressions, one gets

$$g_c(z,v) = g_c^{(1)}(z,v) + g_c^{(2)}(z,v),$$

where

$$\begin{aligned} g_c^{(1)}(z,v) &= c^{-1}B_c(z,v)g^{(1)}(z, \{v+\ln c\}_{q(z)}) - c^{-1}g^{(1)}(z,v), \\ g_c^{(2)}(z,v) &= c^{-1}(B_{e^v}(z,0))^{-1} \sum_{k\in[0,n)} B_{e^{r_k(z)}}(z,0) g_1(V^k z), \end{aligned}$$

if $r_n(z) \le \ln c + v < r_{n+1}(z)$, for some measurable functions $g^{(1)} : Z \times [0, q(\cdot)) \mapsto \mathbb{R}$ and $g_1 : Z \mapsto \mathbb{R}$.

The function $g_c^{(1)}(z,v)$ has the form (4.9) since $c^{-1}B_c(z,v) = 1$. Consider now the function $g_c^{(2)}(z,v)$. Since $r_n(z) = nq(z)$ by (5.1), we have $r_n(z) \le \ln c + v < r_{n+1}(z)$ when $n = [v+\ln c]_{q(z)}$ using (4.7). By using $B_c(z,v) = c$ and (5.1), we obtain that

$$g_c^{(2)}(z,v) = g_1(z)\, e^{-v-\ln c} \sum_{k\in[0,[v+\ln c]_{q(z)})} e^{kq(z)} = g_0(z)\, e^{-v-\ln c}\Big(e^{[v+\ln c]_{q(z)}q(z)} - 1\Big),$$

where $g_0(z) = g_1(z)/(e^{q(z)} - 1)$. Applying (4.7), we get

$$g_c^{(2)}(z,v) = g_0(z)(e^{-\{v+\ln c\}_{q(z)}} - c^{-1}e^{-v}),$$

which has the form (4.9) with $g(z,v) = g_0(z)e^{-v}$. Thus $g_c^{(1)} + g_c^{(2)}$ has the form (4.9). $\Box$

Proposition 5.2 *Let $\{\psi_c\}_{c>0}$ be a cyclic flow on the space $Z \times [0, q(\cdot))$ given by (4.8). Let also $\{j_c\}_{c>0}$ be a 2-semi-additive functional for the flow $\{\psi_c\}_{c>0}$ satisfying (1.10), and choose a version of the Radon-Nikodym derivatives*

$$\frac{d((\sigma \otimes \mathbb{L}) \circ \psi_c)}{d(\sigma \otimes \mathbb{L})}(z,v) \equiv 1$$

which is a cocycle for the flow $\{\psi_c\}_{c>0}$, where $\sigma(dz)$ is a measure on Z and $\mathbb{L}$ denotes the Lebesgue measure on $\mathbb{R}$. Then, the solution to (1.10) is given by (4.11).

PROOF: Example 3.2 shows that $J_c(z,v) = c^{H-1/\alpha} j_c(z,v)$ is a semi-additive functional related to the cocycle $B_c(z,v) = c^{H-1/\alpha} b_c(z,v)$. Since $\psi_c(z,v)$ has a special representation (2.6) with t replaced by $\ln c$, and V and r_n given in (5.1), we can apply Corollary 2.3 to express $\{J_c\}_{c>0}$ by the sum of two functionals. By substituting $J_c(z,v) = c^{H-1/\alpha} j_c(z,v)$ into these expressions, we get

$$j_c(z,v) = j_c^{(1)}(z,v) + j_c^{(2)}(z,v),$$

where

$$\begin{aligned} j_c^{(1)}(z,v) &= c^{-(H-1/\alpha)} B_c(z,v) j^{(1)}(z, \{v + \ln c\}_{q(z)}) - c^{-(H-1/\alpha)} j^{(1)}(z,v), \\ j_c^{(2)}(z,v) &= c^{-(H-1/\alpha)} (B_{e^v}(z,0))^{-1} \sum_{k \in [0,n)} B_{e^{r_k(z)}}(z,0) j_1(V^k z), \end{aligned}$$

if $r_n(z) \le \ln c + v < r_{n+1}(z)$, for some measurable functions $j^{(1)} : Z \times [0, q(\cdot)) \mapsto \mathbb{R}$ and $j_1 : Z \mapsto \mathbb{R}$.

Since $c^{-(H-1/\alpha)} B_c(z,v) = b_c(z,v)$ and b_c is given by (4.10), the function $j_c^{(1)}(z,v)$ has the form of the first two terms of (4.11). Consider now the function $j_c^{(2)}(z,v)$. Observe that $r_n(z) \le \ln c + v < r_{n+1}(z)$ when $n = [v + \ln c]_{q(z)}$, since $r_n(z) = nq(z)$. Since $B_c(z,v) = c^{H-1/\alpha} b_c(z,v)$, $V^n z = z$ and $r_n(z) = nq(z)$, we obtain that

$$j_c^{(2)}(z,v) = j_1(z) c^{-(H-1/\alpha)} e^{-(H-1/\alpha)v} (b_{e^v}(z,0))^{-1} \sum_{k \in [0, [v+\ln c]_{q(z)})} e^{(H-1/\alpha)kq(z)} b_{e^{kq(z)}}(z,0).$$

By using the expression (4.10) of $b_c(z,v)$, we have

$$b_{e^v}(z,0) = b_1(z)^{[v]_{q(z)}} \frac{b(z, \{v\}_{q(z)})}{b(z,0)} = \frac{b(z,v)}{b(z,0)},$$

since $(z,v) \in Z \times [0, q(\cdot))$ and $b_1 \in \{-1, 1\}$. We also get that $b_{e^{kq(z)}}(z,0) = b_1(z)^k$. Hence,

$$j_c^{(2)}(z,v) = j_0(z)(b(z,v))^{-1} e^{-(H-1/\alpha)(v+\ln c)} \sum_{k \in [0, [v+\ln c]_{q(z)})} e^{(H-1/\alpha)kq(z)} b_1(z)^k,$$

where $j_0(z) = j_1(z)b(z,0)$. If $H \neq 1/\alpha$ or $b_1(z) \neq 1$, then $e^{(H-1/\alpha)q(z)}b_1(z) \neq 1$. Hence, by using (4.7) and (4.10),

$$\begin{aligned}
j_c^{(2)}(z,v) &= j(z)(b(z,v))^{-1}e^{-(H-1/\alpha)(v+\ln c)}\Big(e^{(H-1/\alpha))q(z)[v+\ln c]_{q(z)}}b_1(z)^{[v+\ln c]_{q(z)}} - 1\Big) \\
&= j(z)\left(\frac{e^{-(H-1/\alpha)\{v+\ln c\}_{q(z)}}}{b(z,v)}b_1(z)^{[v+\ln c]_{q(z)}} - \frac{e^{-(H-1/\alpha)(v+\ln c)}}{b(z,v)}\right) \\
&= j(z)\left(b_c(z,v)\frac{e^{-(H-1/\alpha)\{v+\ln c\}_{q(z)}}}{b(z,\{v+\ln c\}_{q(z)})} - c^{-(H-1/\alpha)}\frac{e^{-(H-1/\alpha)v}}{b(z,v)}\right),
\end{aligned}$$

where $j(z) = j_0(z)/(e^{(H-1/\alpha)q(z)}b_1(z) - 1)$. Thus $j_c^{(2)}(z,v)$ with $H \neq 1/\alpha$ or $b_1(z) \neq 1$ has the form of the first two terms of (4.11) (as did $j_c^{(1)}(z,v)$). If $H = 1/\alpha$ and $b_1(z) = 1$, then

$$j_c^{(2)}(z,v) = j_0(z)(b(z,v))^{-1}\sum_{k\in[0,[v+\ln c]_{q(z)})} 1 = j_0(z)(b(z,v))^{-1}[v+\ln c]_{q(z)},$$

which has the form of the last term of (4.11). $\square$

References

Bingham, N. H., Goldie, C. M. & Teugels, J. L. (1987), *Regular Variation*, Cambridge University Press.

Cornfeld, I. P., Fomin, S. V. & Sinai, Y. G. (1982), *Ergodic Theory*, Springer-Verlag.

Hardin Jr., C. D. (1981), 'Isometries on subspaces of L^p', *Indiana University Mathematics Journal* **30**, 449–465.

Kolodyński, S. & Rosiński, J. (2002), 'Group self-similar stable processes in $\mathbb{R}^d$', *Journal of Theoretical Probability* **16**(4), 855–876.

Kubo, I. (1969), 'Quasi-flows', *Nagoya Mathematical Journal* **35**, 1–30.

Kubo, I. (1970), 'Quasi-flows II: Additive functionals and TQ-systems', *Nagoya Mathematical Journal* **40**, 39–66.

Pipiras, V. & Taqqu, M. S. (2002*a*), 'Decomposition of self-similar stable mixed moving averages', *Probability Theory and Related Fields* **123**(3), 412–452.

Pipiras, V. & Taqqu, M. S. (2002*b*), 'The structure of self-similar stable mixed moving averages', *The Annals of Probability* **30**(2), 898–932.

Pipiras, V. & Taqqu, M. S. (2003*a*), Dilated fractional stable motions. To appear in *The Journal of Theoretical Probability.*

Pipiras, V. & Taqqu, M. S. (2003*b*), Identification of periodic and cyclic fractional stable motions, Preprint.

Pipiras, V. & Taqqu, M. S. (2003*c*), Integral representations of periodic and cyclic fractional stable motions, Preprint.

Pipiras, V. & Taqqu, M. S. (2003*d*), Stable stationary processes related to cyclic flows. To appear in *The Annals of Probability.*

Rosiński, J. (1995), 'On the structure of stationary stable processes', *The Annals of Probability* **23**, 1163–1187.

Samorodnitsky, G. & Taqqu, M. S. (1994), *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance*, Chapman and Hall, New York, London.

Surgailis, D., Rosiński, J., Mandrekar, V. & Cambanis, S. (1998), On the mixing structure of stationary increment and self-similar $S\alpha S$ processes, Preprint.

Zimmer, R. J. (1984), *Ergodic Theory and Semisimple Groups*, Birkhäuser, Boston.

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